Inner Product Spaces

1. Notation and Properties:

Let v and w be vectors in a vector space V.
The inner product of v and w is denoted by
$\langle v, w \rangle$.

Properties:
Let u, v, w be vectors in V.
Let r be a scalar in $\mathbb{R}$.

1. $\langle v, w \rangle = \langle w, v \rangle$
2. $r\langle v, w \rangle = \langle rv, w \rangle = \langle v, rw \rangle$
3. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

E.g. 1 Determine whether or not R^2 is an inner product space if, for $v = [v_1, v_2]$ and $w = [w_1, w_2]$, we define
$\langle v, w \rangle = 2v_1w_1 + 5v_2w_2$

Solution

We have to verify that $\langle v, w \rangle$ satisfies all 4 properties.

1. $\langle v, w \rangle = 2v_1w_1 + 5v_2w_2$

$\langle w, v \rangle = 2w_1v_1 + 5w_2v_2$

$= \langle v, w \rangle$

2. $r\langle v, w \rangle = r(2v_1w_1 + 5v_2w_2)$

$= 2rv_1w_1 + 5rv_2w_2$

$$\langle rv, w\rangle = 2(rv_1)w_1 + 5(rv_2)w_2$$
$$= 2rv_1w_1 + 5rv_2w_2$$

$$\langle v, rw\rangle = 2v_1(rw_1) + 5v_2(rw_2)$$
$$= 2rv_1w_1 + 5rv_2w_2$$

3. $\langle u, v+w\rangle = 2u_1(v_1+w_1) + 5u_2(v_2+w_2)$
$$= 2u_1v_1 + 2u_1w_1 + 5u_2v_2 + 5u_2w_2$$

$$\langle u,v\rangle = 2u_1v_1 + 5u_2v_2$$
$$\langle u, w\rangle = 2u_1w_1 + 5u_2w_2$$

$$\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$$

4. $\langle v, v\rangle = 2v_1^2 + 5v_2^2$

We see that $\langle v,v\rangle \geq 0$ and is equal to 0 iff $v=0$.

$\therefore$ It is an inner-product space.

2. Magnitude of a vector:

Let V be an inner-product space. Let v be a vector in V. Then, $\|v\| = \sqrt{\langle v,v\rangle}$. Furthermore, $\|v\|^2 = \langle v,v\rangle$.

Note: $\|rv\| = |r|\|v\|$

Proof:

$$\|rv\|^2 = \langle rv, rv\rangle$$
$$= r^2\langle v,v\rangle$$
$$= r^2\|v\|^2$$

$$\|rv\| = |r|\,\|v\|$$

QED

3. Schwartz and Triangle Inequality:

1. $\cos\theta = \dfrac{\langle v,w\rangle}{\|v\|\|w\|}$

2. $|\langle v,w\rangle| \leq \|v\|\|w\|$

3. $\|v+w\| \leq \|v\| + \|w\|$

4. Iff $\langle v,w\rangle = 0$, then v and w are orthogonal.